Packing P´s - Solution

As each P-pentomino has area 5 and the suitcase has a total area of n^2, an obvious upper bound on the number of P-pentonimos is $\lfloor n^2/5 \rfloor$. With the exception of $n = 5$, we claim that this bound can always be attainedthe other hand, suppose there is a packing with 5 P-pentominos. As the entire suitcase must then be tiled, each corner is covered by a P-pentomino. Label these 4 P-pentominos 1 through 4. Any P-pentomino that covers a corner square must also cover the square diagonally next to the corner square, regardless of how the P-pentomino is oriented. Thus, we know that in any packing with 5 P-pentominos, we have the situation in Figure 1(b). Even in this partial description of the packing, we can no longer fit a fifth P-pentomino, and so no packing with 5 P-pentominos exists for $n = 5$. The optimal answer for $n = 5$ is therefore 4.

(a) A tiling with 4 P-pentominos.

(b) A partial cover that any tiling with 5 P-pentominos has.

Figure 1: The exceptional case $n = 5$.

To show that the suitcase can be packed with $\lfloor n^2/5 \rfloor$ P-pentominos for any $n \neq 5$ we give an inductive construction. The key to the induction step is the following construction.

Lemma 1. *For any $n \in \mathbb{N}$ with $n \neq 1$ and $n \neq 3$, if an $n \times n$ square can be packed with m P-pentominos, then an $(n+10) \times (n+10)$ square can be packed with $m + 4n + 20$ P-pentominos.*

Proof. Split the $(n+10) \times (n+10)$ square into an $n \times n$ square, a 10×10 square, and two $10 \times n$ rectangles (one of the rectangles is actually $n \times 10$, but rotation does not change the problem of tiling). Note that we can combine two P-pentominos to tile a 2×5 rectangle, so that we can perfectly tile the 10×10 square with 20 P-pentominos. Also, using these 2×5 rectangles, we can tile a 2×10 and a 5×10 rectangle perfectly. Now, if n is even, we can tile both $10 \times n$ rectangles perfectly with $2n$ P-pentominos each, by stacking $n/2$ tilings of a 2×10 rectangle. On the other hand, as $n \neq 1$ and $n \neq 3$, if n is odd then $n \geq 5$. Thus we can perfectly tile both $10 \times n$ rectangles with $2n$ P-pentominos each by starting with a 5×10 rectangle and adding the $(n-5)/2$ remaining 2×10 rectangles on top. By assumption the $n \times n$ square can be packed with m P-pentominos, so that combining the packings yields a packing of $m + 4n + 20$ P-pentominos in the $(n+10) \times (n+10)$ square. □

As $\lfloor (n+10)^2/5 \rfloor = \lfloor n^2/5 \rfloor + 4n + 20$, Lemma 1 suffices as the induction step. To complete the induction, we only need to check the base cases $n \in \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 15\}$, skipping 5 as we know a 5×5 square cannot be tiled perfectly by P-pentominos. All cases except $n = 15$ are left to the reader; a useful tool is the 2×5 rectangle to reduce the larger n to smaller areas to tile. As the case $n = 15$ is somewhat exceptional since $n = 5$ cannot be tiled perfectly by P-pentominos, we have included a possible construction in Figure 2.

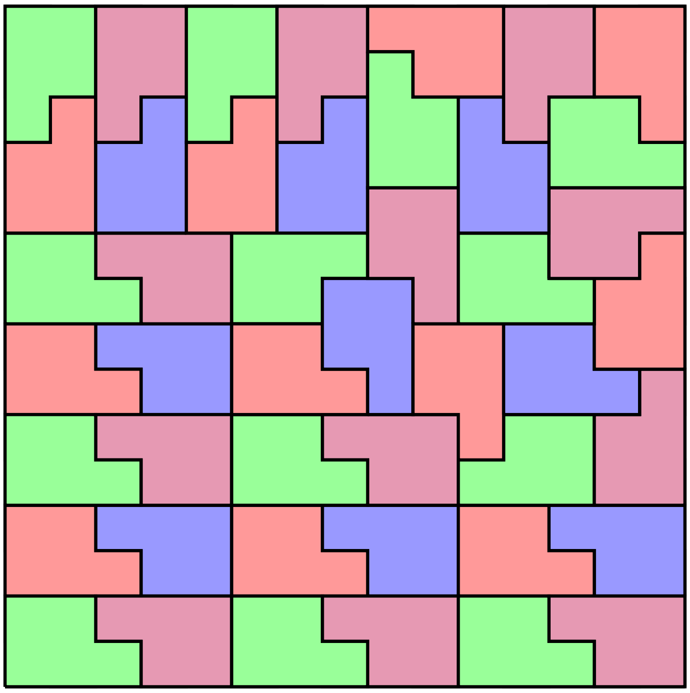

Figure 2: The case $n = 15$.