Cubic Counting - Solution

Write $P(x) = x^3 + Ax^2 + Bx + C$. Of course we can use generic methods for finding roots of polynomials. However, as P has a specific form, we can do something simpler. As P is of degree 3, has no double roots, is continuous, and has $\lim_{x\to\infty} P(x) = \infty$ and $\lim_{x\to-\infty} = -\infty$, P either has 1 or 3 real roots. We note that P can only have 3 roots if the derivative of P is negative at some point. Otherwise P is non-decreasing and thus can only have a single root, as P has no roots of multiplicity greater than 1. If the derivative

$$3x^2 + 2Ax + B$$

is negative at some point, it must have two zeroes, corresponding to a local maximum and local minimum of P. If P has three distinct roots, the derivative at the first and third roots will be positive and the derivative is negative at the second root. Hence, if P has three roots, the local maximum and local maximum have different signs. On the other hand, if the local maximum and local minimum exist and have different signs then P must have 3 roots. Hence it suffices to compute the local minimum and maximum by plugging in the zeroes of the derivative $3x^2 + 2Ax + B$. Its discriminant is $D = 4A^2 - 12B$, and we saw if $D \leq 0$ then there is 1 root. Otherwise, check if

$$P\left(\frac{-2A + \sqrt{D}}{6}\right) P\left(\frac{-2A - \sqrt{D}}{6}\right) < 0.$$

If so, P has 3 roots and otherwise P has 1. To prevent floating point errors, the calculations can be performed only in integers, by rewriting slightly, as we are only interested in the sign of the product above, not its value. However, this was not required to pass all test cases.