MATHEMATICS DEPARTMENT

"A computer is the mathematicians best friend"

μ - Games
Mathematics Utrecht

February 2024

Rules:

The idea of this event is to gap the bridge between mathematics and programming. When working on these exercises, we hope the participant will get a better understanding of the underlying mathematical concepts. You will not be required to do a lot of difficult programming. With array manipulation and basic functionality, you should be able to solve all the exercises.

When working on these exercises, you must conform to the following rules.

- You are allowed to work in groups of maximum 4 persons.
- You will have 3 hours to solve the problems.
- For the problems, you can use the default mathematics library of your programming language (for example *import math* in Python).
- You cannot look up any computer code that may help you with solving the problem.

After 3 hours, the solutions to the exercises will be discussed. To check your own solution, one can go to the website `http://clover.science.uu.nl/dj`.

The function *pow(n,p,m)* in Python may prove useful; it provides an efficient way to compute $n^p \pmod{m}$.

Problem 1: Cubic Counting

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Difficulty: ★ ☆ ☆ ☆ ☆
Key words: Algebra
How many distinct real roots does the cubic

$$x^3 + Ax^2 + Bx + C$$

have? It is guaranteed that the cubic has no double or triple roots.

Input

- Three space-separated integers $-10^5 \leq A, B, C \leq 10^5$, the coefficients of the cubic.

Output

- A single integer, the number of real roots of the cubic.

Examples

Input	Output
0 0 1	1

Input	Output
-3 -2 1	3

Problem 2: Generating Goldbach

February 2024

Difficulty: ★ ★ ☆ ☆ ☆
Key words: Number Theory

Goldbach's Conjecture is a famous conjecture that states that every even number greater than 2 is the sum of two primes. For instance, $4 = 2 + 2$ and $16 = 3 + 13$. Given an even number n greater than 2, can you find two primes that sum to n?

Input

- A single even integer $2 < n < 10^5$, the desired sum.

Output

- Two space-separated primes that sum to n. If there are multiple answers, you may output any one of them.

Examples

Input	Output
6	3 3

Input	Output
16	3 13

Problem 3: Power Functions

February 2024

Difficulty: ★ ★ ★ ☆ ☆
Key words: Set theory

This is an interactive problem.

Consider the set $[n] = \{1, ..., n\}$ and an order preserving function

$$f : \mathcal{P}([n]) \to \mathcal{P}([n]),$$

where $\mathcal{P}([n])$ denotes the power set of $[n]$. Then, an order preserving function f is a function such that for $A \subseteq B \subseteq [n]$, $f(A) \subseteq f(B)$. It is known that such a function always has a fixed point. However, we are now interested in finding such a fixed point. The computer knows some secret function and you can request values of f by giving it some (ordered) subset of $[n]$. The computer responds with the value of f. Find a fixed point in at most $n + 3$ guesses.

A non-empty set is given by one integer with digits $1 \leq d \leq 9$ sorted from smallest to largest digit. The digits in each number represent the elements present in the set. The empty set is specified by a 0.

Input

- One line with as input $1 \leq n \leq 9$ the size of the set.
- Then, for each set you guess, the program replies with the result of the function f. The program should terminate when it receives an output that matches the input.

Output

- A subset of $[n]$ you want to guess.

Examples

In the below examples, we have put each in/output on its own line, so that it is clear in what order the in and output is given.

Input	Output
4	
	1234
0	
	0
0	

Input	Output
2	
	1
12	
	12
12	

Problem 4: Elegant Expectations

Difficulty: ★ ★ ☆ ☆ ☆
Keywords: Probability, Financial mathematics.

In probability theory there exists a certain class of sequences of random variables $(Y_n)_{n\geq 0}$ that satisfy the condition

$$\mathbb{E}[X_{n+1} \mid (X_0, X_1, \dots, X_n)] = X_n, \tag{1}$$

where the function $\mathbb{E}[X_{n+1} \mid (Y_0, Y_1, \dots, Y_n)]$ indicates the expectation value of the random variable X_{n+1}, given all the prior information up until the index $n \in \mathbb{N}$. We are interested in the sequence of random variables

$$S_n = \gamma Y_n + Y_{n-1}, \tag{2}$$

where $(Y_n)_{n\geq 0}$ satisfies

$$\mathbb{E}[Y_{n+1} \mid (Y_1, Y_2, \dots, Y_n)] = aY_n + bY_{n-1}, \qquad a + b = 1. \tag{3}$$

Input

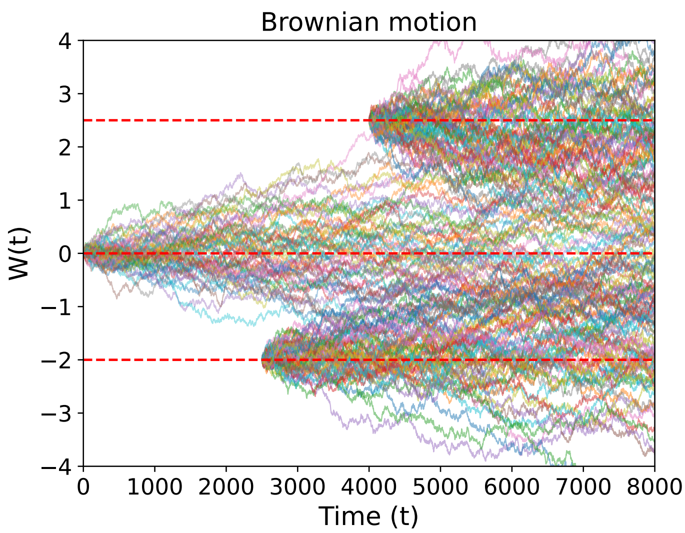
- Line containing a float representing $0 < a < 1$
- Line containing a float representing $0 < b < 1$
- Line containing an integer representing $0 < k < 10^5$
- Line containing an integer representing $0 < \mathbb{E}[Y_0] < 10^5$
- Line containing an integer representing $0 < \mathbb{E}[Y_1] < 10^5$

Output

- One line containing a float representing the value γ for which $(S_n)_{n\geq 0}$ satisfies equality (1) w.r.t. information $(Y_0, Y_1, \dots, Y_n)$.
- One line containing the expectation value $\mathbb{E}[S_k]$

Examples

Input	Output
0.31415 0.68585 10 2 3	1.458045 6.37413



Problem 5: Packing P´s

Difficulty: ★ ★ ★ ★ ☆
Key words: Packing, Combinatorics

A certain old man from Spain has requested your help packing his luggage. In one of his square suitcases of side length n, he would like to fit as many boxes as possible. The boxes are shaped like a P-pentomino, as shown below. Strangely, all the boxes smell like chocolate, but you decide to not pay too much attention to the smell. Given that the suitcase is a square with sides of length n, how many boxes can you fit into the suitcase without overlap? Mirroring and rotating the boxes is allowed.

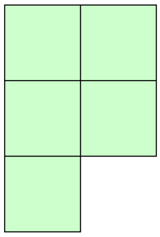


Figure 1: A P-pentomino.

Input

- A single integer $1 \leq n < 10^4$, the side length of the old man's suitcase.

Output

- A single integer, the maximal number of P-pentomino shaped boxes that fit.

Examples

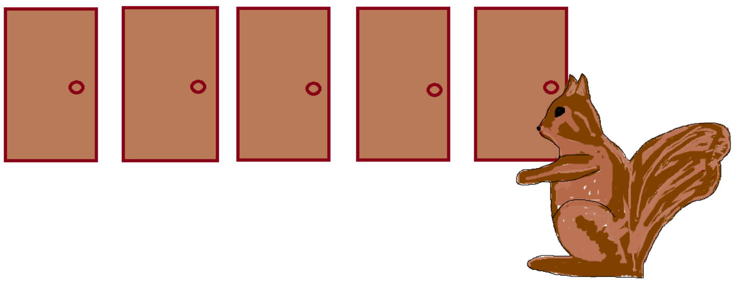
Input	Output
3	1

Input	Output
4	3

Problem 6: Hidden Hazelnuts

Difficulty: ★ ★ ★ ★ ☆
Key words: Markov chain, Abelian sandpile model

Sasha the squirrel has gathered a lot of hazelnuts (in fact, infinitely many!) throughout the last months, but now Sasha has to hide them away, as his squirrel friends are coming to visit and they will otherwise steal all of his hazelnuts. To hide the hazelnuts, Sasha has put N cabinets inside his home:



For starters, he puts one hazelnut inside every cabinet. Then, he will go through the following step infinitely many times:

- Put a new hazelnut in a uniformly random cabinet
- If the cabinet has now 3 hazelnuts inside it, it is too full. Therefore Sasha grabs 2 of the hazelnuts and puts one in the cabinet left and one in the cabinet right of this cabinet. If the cabinet is at the end of the line of cabinets, he will throw away one nut
- If now any of the other cabinets are too full as well, we will redistribute again. The redistributing will take place until none of the cabinets are too full anymore.

Now we call a distribution of hazelnuts in the cabinets *recurrent* when the distribution will keep occurring when we keep adding new hazelnuts for an infinite amount of times. We call a distribution *stable* if none of the cabinets is too full, i.e. just before we are in the part of the process just before we will add a new hazelnut.
Sasha wants to know if a given distribution is recurrent and stable.

Input

- An integer N, the size of the chain
- N space-seperated heights of the sites $h_1, ..., h_N$, where all of the heights are bigger or equal to 1, i.e. $h_i \geq 1$ for all i

Output

- The string "Yes" if the state is stable-recurrent, "No" if the state is not recurrent or non-stable.

Examples

Input	Output
2 2 2	Yes

Input	Output
3 1 1 2	No

Input	Output
8 1 1 1 1 1 1 1 1	No

Problem 7: Grazing Goats

Difficulty: ★ ★ ★ ★ ★
Key words: Geometry

A herd of n goats are grazing in an infinitely large field near Utrecht. Goat i is connected to its own post by a rope of finite length r_i, so that each goat can graze all the grass in a circle of radius r_i around its post. To ensure each goat has something to eat, the posts are placed in such a way that each goat at least has some section of the field (of positive area) to itself. Given all rope lengths and the location of each post, what is the total area that the goats can graze together?

Input

- The first line contains a single integer $1 \leq n \leq 100$, the number of goats.
- Next follow n lines describing a goat, each containing three space-separated integers $1 \leq r_i \leq 10^5$, $-10^5 \leq x_i, y_i \leq 10^5$, the length of the rope and the position of the post, respectively.

Output

- The total area that can be grazed by the goats. Your answer should have a relative error of at most 10^{-6}.

Examples

Input	Output
1 1 0 0	3.14159265

Input	Output
3 4 3 0 4 -3 0 3 0 6	120.77932148