Problem 4: Elegant Expectations - Solution

We first want to find the γ value for which the sequence of random variables $(S_n)_{n\geq 0}$ fullfills requirement (1). When a process satisfies this requirement, it is called a martingale. We calculate:

$$\begin{aligned}
\mathbb{E}[S_{n+1} \mid (Y_0, Y_1, \dots, Y_n)] &= \mathbb{E}[\gamma X_{n+1} + X_n \mid (Y_0, Y_1, \dots, Y_n)] && (1)\\
&= \gamma \mathbb{E}[Y_{n+1} \mid (Y_0, Y_1, \dots, Y_n)] + \mathbb{E}[Y_n \mid (Y_0, Y_1, \dots, Y_n)] && (2)\\
&= \gamma\,(aY_n + bY_{n-1}) + Y_n && (3)\\
&= (\gamma a + 1)Y_n + \gamma b Y_{n-1} && (4)\\
& && (5)
\end{aligned}$$

If we want to satisfy equality (1), we need this expectation value to equal $S_n = \gamma Y_n + Y_{n-1}$. The only γ value for which this requirement is satisfied is $\gamma = 1/b$ (or equivalently $\gamma = \frac{1}{1-a}$).
Note that when a system has property (1), we can iterate this procedure as follows:

$$\begin{aligned}
\mathbb{E}[Y_i] &= \mathbb{E}[\mathbb{E}[Y_i \mid (Y_0, Y_1, \dots, Y_{i-1})]] && (6)\\
&= \mathbb{E}[aY_{i-1} + bY_{i-2}] && (7)\\
&= a\mathbb{E}[Y_{i-1}] + b\mathbb{E}[Y_{i-2}] && (8)
\end{aligned}$$

One can use this iteration to get the expected value $\mathbb{E}[X_k]$. With this information, one is able to extract the expected value $\mathbb{E}[S_k]$.

An alternative way is to use property (1). We get:

$$\begin{aligned}
\mathbb{E}[S_k] &= \mathbb{E}[\mathbb{E}[S_k \mid (Y_0, Y_1, \dots, Y_{k-1})]] && (9)\\
&= \mathbb{E}[S_{k-1}] && (10)\\
&\vdots && (11)\\
&= \mathbb{E}[S_1] && (12)\\
&= \mathbb{E}[\frac{Y_1}{b} + Y_0] && (13)\\
&= \frac{\mathbb{E}[Y_1]}{b} + \mathbb{E}[Y_0]. && (14)
\end{aligned}$$

Al this values are known and therefore the solution is complete.