MATHEMATICS DEPARTMENT

"A computer is the mathematicians best friend"

μ - Games
Mathematics Utrecht

April 2022

Rules:

The idea of this event is to gap the bridge between mathematics and programming. When working on these exercises, we hope the participant will get a better understanding of the underlying mathematical concepts used. You will not be required to do a lot of difficult programming. With array manipulation and basic functionality you should be able to solve all the exercises.

When working on these exercises, you must conform to the following rules.

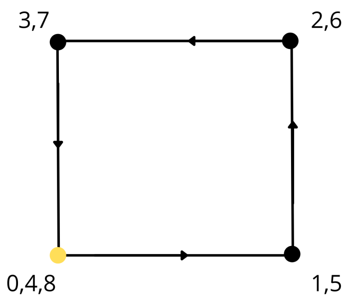
- You are allowed to work in groups of maximum 4 persons.
- You will have three hours time.
- You can use no software packages.
- You can not look up any computer code that may help you solve the problem online.

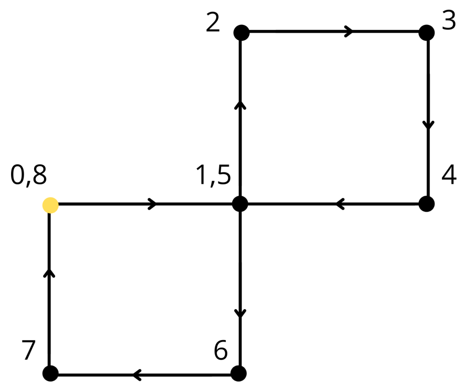
After the three hours, the solution to the exercises will be discussed and the solution has to be posted on the website *http://clover.science.uu.nl/dj*.

Problem 1: Robot Walk

Difficulty: ★ ☆ ☆ ☆ ☆
Keywords: Counting, Geometry

A robot wants to go out and take a walk of exactly n steps. After the n steps, he wants to be right back where he started. This robot walks in a peculiar way. He starts his path facing north. Then he either turns 90° to the left or to the right, upon which he takes a step. Then he again turns 90° to the left or to the right, upon which he takes a second step. This process repeats until he took n steps. An example of two paths of 8 steps looks like this:





Now the question is, how many of such walks of n steps are there that end at the starting point?

Input

- The number of steps $1 \leq n \leq 2^{63}$.

Output

- The number of paths the robot can take

Examples

Input	Output
4	16

Input	Output
8	36

Problem 2: Odd Sets

Difficulty: ★ ☆ ☆ ☆ ☆
Keywords: Algebra, finite fields

Let S be a set with n elements, and 2^S denotes the power set.
Consider a family of distinct sets $S_i \in 2^S$, $i = 1 \dots m$.

What is the maximum number of sets m, such that:

1. Each pair of distinct sets have an even number of elements in common, and
2. There are no sets with even number of elements?

Input

- One line containing integer $1 \leq n \leq 1000$, the number of elements.

Output

- One line containing m, the maximum number of distinct sets satisfying the criteria (1) and (2).

Example

Input	Output
3	3

Explanation: $S = \{1, 2, 3\}$, then all possible subsets are: $2^S = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}\ \{3\}, \{\}\}$
One can see that $S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}$ is the largest family of subsets, hence m=3.

Problem 3: Unstable Matrices

Difficulty: ★ ★ ★ ☆ ☆
Keywords: Linear Algebra

We define the matrix F_n as

$$F_n = \begin{pmatrix} n & n-1 & n-2 & \dots & 2 & 1 \\ n-1 & n-1 & n-2 & \dots & 2 & 1 \\ 0 & n-2 & n-2 & \dots & 2 & 1 \\ \vdots & 0 & \ddots & \ddots & \vdots & 1 \\ \vdots & \vdots & \dots & 2 & 2 & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}.$$

One can show analytically that the determinant of F_n is given by

$$det(F_n) = 1.$$

This matrix is a good test matrix for algorithms that find eigenvalues. This is because this matrix is very sensitive to changes in the entries. If you perturb the $(1, n)$ element to be $1 + \epsilon$, the determinant of this matrix changes from 1 to $(n-1)!\epsilon$, which is very difficult to handle for a computer.

The characteristic polynomial of the matrix F_n is given by

$$p_n = det(F_n - \lambda \mathbb{I}).$$

We denote the coefficients $(a_i)_{i=1}^{i=n}$ of this polynomial by

$$p_n = \sum_{i=0}^{n} a_i \lambda^i.$$

In this exercise we are interested in the coefficients of the characteristic polynomial of F_n.

Input

- One line with the integer $1 \leq n \leq 1000$, which represents the size of matrix F_n.
- $\lceil n/2 \rceil$ lines, each with an integer representing coefficient a_0 till $a_{\lceil n/2 \rceil - 1}$ of the characteristic polynomial of F_n.

Output

- $n - \lfloor n/2 \rfloor$ lines, each with an integer representing coefficient $a_{\lfloor n/2 \rfloor + 1}$ till a_n of the characteristic polynomial of F_n.

Example

Input	Output
3	6
1	-1
-6	

Problem 4: Famous Function

Difficulty: ★ ★ ★ ★ ☆
Keywords: Integration

A famous researcher has earned a Nobel prize by doing research involving the density function σ_t given by

$$\sigma_t(x) = \frac{1}{2\pi t}\sqrt{(4t - x^2)_+}dx, \tag{1}$$

for x $\in \mathbb{R}$. Here $x_+ = \max\{x, 0\}$.

We are interested in

$$m_k = \int_{\infty}^{-\infty} x^k \sigma_t(x)\, dx. \tag{2}$$

Input

- One line with a positive integer $1 \leq k \leq 1000$.
- One line with a positive integer $1 \leq t \leq 1000$.

Output

- An integer representing m_k.

Example

Input	Output
2 3	3

Problem 5: Limiting Behaviour

Difficulty: ★★★★★
Keywords: Dynamical Systems

Consider a system of ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = diag(\mathbf{x})(1 - (2\mathbf{A} + \mathbf{I})\mathbf{x}), \tag{3}$$

$$\mathbf{x}(0) = \mathbf{x}_0, \tag{4}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{A}$ is the adjacency matrix of a *circle graph* G with n vertices. That is, G is a connected graph in which every vertex has two neighbours.

It is known that, asymptotically, this system always arrives at a stable fixed point, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$ (*i.e.* the limit can be taken element-wisely), however $\mathbf{x}^*$ is not unique and may depend on the initial conditions.

How many distinct stable fixed points $\mathbf{x}^*$ can the system arrive at if $\mathbf{x}_0 \in (0,1)^n$ and x_0 is no fixed point?

Input

- One line containing a positive integer $3 \leq n \leq 1000$ representing the size of the graph.

Output

- One line containing the number of different fixed points we can reach. This must be an integer.

Example

Input	Output
3	3