Problem 3: Unstable Matrix Solution

The Frank matrix is defined as

$$F_n = \begin{pmatrix} n & n-1 & n-2 & \dots & 2 & 1 \\ n-1 & n-1 & n-2 & \dots & 2 & 1 \\ 0 & n-2 & n-2 & \dots & 2 & 1 \\ \vdots & 0 & \ddots & \ddots & \vdots & 1 \\ \vdots & \vdots & \dots & 2 & 2 & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

We define $p_n = det(F_n - \lambda \mathbb{I}_n)$. By taking the Laplace expansion, we obtain

$$\begin{aligned} p_n &= (n-\lambda)p_{n-1} - (n-1)\begin{vmatrix} n-1 & n-1 & \dots & 1 \\ n-2 & n-2-\lambda & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)p_{n-1} - (n-1)\left[\begin{vmatrix} n-1 & n-2 & \dots & 1 \\ n-2 & n-2-\lambda & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} - p_{n-1}\right] \end{aligned}$$

The terms in the brackets we can write out as

$$\begin{aligned} &\begin{vmatrix} n-1 & n-2 & \dots & 1 \\ n-2 & n-2-\lambda & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} - p_{n-1} \\ &= (n-1)p_{n-2} - (n-2)\begin{vmatrix} n-2 & n-3 & \dots & 1 \\ n-2 & n-3-\lambda & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} - (n-1-\lambda)p_{n-2} + (n-2)\begin{vmatrix} n-2 & n-3 & \dots & 1 \\ n-2 & n-3-\lambda & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} \\ &= \lambda p_{n-2}. \end{aligned}$$

Combined, we get the recursive relation

$$p_n = (1-\lambda)p_{n-1} - (n-1)\lambda p_{n-2}.$$

We can use this to show

$$p_n(\lambda^{-1}) = (-1)^n \lambda^{-n} p_n(\lambda).$$

Hence, the coefficient spectrum is palindromic when n is even and anti-palindromic when n is odd.