Problem 4: Famous Function Solution

We can calculate

$$\begin{aligned}
\int x^k \sigma_t(dx) &= \int x^k \frac{1}{2\pi t}\sqrt{(4t-x^2)_+}\,dx \\
&= \int x^k \frac{1}{2\pi t} t^{1/2}\sqrt{(4-(x/\sqrt{2})^2)_+}\,dx \\
&= \int x^k t^{k/2} \frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx \\
&= t^{k/2}\int x^k \frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx \\
&= t^{k/2}\int x^k \sigma_1\,dx.
\end{aligned}$$

Note that for odd k, this integral equals zero. Now look at $2k$ for $k \in \mathbb{N}_{>0}$. We can calculate $\int x^k \sigma_1\,dx$ by making the substitution $x = 2\sin\theta$. This gives

$$\begin{aligned}
\int x^{2k}\sigma_1\,dx &= \int \frac{1}{2\pi} 2^{2k}\sin^{2k}\theta(4-4\sin^2\theta)^{1/2}\cdot 2\cos\theta\,d\theta \\
&= \frac{1}{2\pi}\int 2^{2k}\sin^{2k}\theta\cdot 2\cos\theta\cdot 2\cos\theta\,d\theta \\
&= \frac{1}{2\pi}\int 2^{2k+2}\sin^{2k}\theta\cos^2\theta\,d\theta \\
&= \frac{2^{2k+1}}{\pi}\left[\int \sin^{2k}\theta\,d\theta - \int \sin^{2k+2}\theta\,d\theta\right] \\
&= 2^{2k+1}\left[\prod_{l=1}^{k}\frac{2l-1}{2l} - \prod_{l=1}^{k+1}\frac{2l-1}{2l}\right] \\
&= \frac{2^{2k+1}}{2k+2}\prod_{l=1}^{k}\frac{2l-1}{2l} \\
&= \frac{2^{k+1}}{2k+2}\cdot\frac{1}{k!}\cdot\frac{(2k)!}{\prod_{l=1}^{k}2l} \\
&= \frac{2}{2k+2}\cdot\frac{(2k)!}{k!k!} \\
&= \frac{1}{k+1}\binom{2k}{k} \\
&= C_k.
\end{aligned}$$

Combining, we have

$$\begin{aligned}
\int x^k\sigma_t(dx) &= t^{k/2}\int x^k\sigma_1(dx) \\
&= t^{k/2}C_{k/2}.
\end{aligned}$$

We can find C_k by using the recursion

$$\begin{aligned}
C_{k+1} &= \frac{2(2k+1)}{k+2}C_k \\
C_1 &= 1.
\end{aligned}$$