Problem 4: Sturm-Liouville Problem - Solution

To solve this problem, one heavily exploits the information already given to you in the problem statement. Let us consider the following differential operator.

$$L = \frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^2\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

We note that the eigenfunctions are polynomials of different degrees i.e. p_n. So, we can use x^n to find eigenvalue of p_n.

$$Lx^n = -n(n+1)x^n + n(n-1)x^{n-2}$$

So, we must have $Lp_n = -n(n+1)p_n$. Comparing the x^d terms on both sides of this equation, we see

$$-d(d+1)a_d + (d+2)(d+1)a_{d+2} = -n(n-1)a_d$$

Thus, we get $a_d = \frac{(d+2)(d+1)}{d(d+1)-n(n-1)}a_{d+2}$. From this recurrence relation, we can now recover the desired polynomial.