Problem 6: Mischievous Moles - Solution

We model the carrot switching by the moles as a Markov process, where the states S are all possible carrot permutations that can be achieved through switching of the initial configuration, and the transitions are induced by the mole tunnels. An example with 3 carrots and tunnels $(1,2)$ and $(2,3)$ can be seen in Figure 1. In terms of Markov processes, we are now interested in the expected first return time of the initial carrot configuration.

From the theory of Markov processes, we have a theorem that states that if a Markov process is finite and has a stationary distribution π, then the expected first return time of any state x is $\frac{1}{\pi_x}$. Clearly the induced Markov process is finite as it has at most $n!$ states, so we only need to find a suitable stationary distribution. The key observation is that, in the induced Markov process, any transition is reversed by performing the same tunnel switch again. Hence, any edge in the graph of the Markov process has the same probability on the flipped edge, as can be seen in Figure 1. Hence, the total incoming probability to a vertex is the same as the total outgoing probability, which by definition is simply 1. Hence, a uniform distribution is stationary and it turns out that the stationary distribution in this case is just $\pi = (\frac{1}{|S|}, \frac{1}{|S|}, \ldots, \frac{1}{|S|})$. Hence, the expected first return time of the initial state is $|S|$; the number of possible carrot configurations that can be achieved by switching carrots through the tunnels.

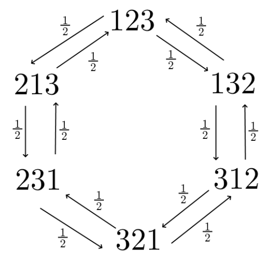

Figure 1: Example for tunnels $(1,2)$ and $(2,3)$.

All that remains is to find $|S|$. Let $G = (V, E)$ denote the "original" graph, with V being the n carrot positions and E all the tunnels between them.

First, suppose that G is connected. Then we can achieve all $n!$ possible carrot permutations by only using the tunnels; take any permutation and only consider the tunnels in some spanning tree of G. Then there is at least one leaf, and we can first move the carrot that corresponds to that spot in the desired permutation there by swapping it with carrots on its path to the leaf. Next, we never have to move this carrot again, as it is in the correct position and is in a leaf, so no other carrot needs to travel "past" it. Hence, we can ignore the carrot and without loss of generality, remove it. Then we can iterate this until all carrots have been moved to their desired position.

Finally, if G is not connected, denote the vertices of its connected components by $V_1, V_2, \ldots, V_k$. Clearly no carrots can switch between spots in V_i and V_j for $i \neq j$, and within V_i, we saw that all $|V_i|!$ permutations can be achieved. As switches in all these connected components are completely independent of each other, the total number of possible permutations that we can achieve (and hence the expected value of T) is precisely

$$|S| = |V_1|!|V_2|! \cdots |V_k|!$$

We can find the size of the connected components from any connected components finding algorithm, such as by using a flood fill with BFS.