MATHEMATICS DEPARTMENT

"A computer is the mathematicians best friend"

μ - Games
Mathematics Utrecht

December 2022

Rules:

The idea of this event is to gap the bridge between mathematics and programming. When working on these exercises, we hope the participant will get a better understanding of the underlying mathematical concepts. You will not be required to do a lot of difficult programming. With array manipulation and basic functionality, you should be able to solve all the exercises.

When working on these exercises, you must conform to the following rules.

- You are allowed to work in groups of maximum 4 persons.
- You will have 3 hours time.
- You cannot use import statements except:
  - import heapq
  - import math
- You cannot look up any computer code that may help you with solving the problem.

After 3 hours, the solutions to the exercises will be discussed. To check your own solution, one can go to the website `http://clover.science.uu.nl/dj`.

Problem 1: Bread Mania

Difficulty: ★ ★ ☆ ☆ ☆
Keywords: Topology, Euler Characteristic

Your friend has a special collection of breads. He has classified these by the number of holes the bread contains. The simplest in his collection is a normal bun with no holes. Then, one gets donuts and bagels with one hole. Then, bread shaped in the form of an eight with two holes. Then, pretzels with three holes; et cetera.
Now, your friend frequently has the following problem: bread traders do not tell one how many holes the bread they sell has. Instead, they give a triangulation of the surface of the bread. Thus, your friend comes to you for help. Can you tell him based on the number of vertices, edges and faces in the triangulation how many holes the piece of bread has?
Hint: a well known invariant of a surface S is the Euler characteristic $\chi(S)$. To calculate this, one considers an arbitrary triangulation (or even more generally a planar decomposition) and takes the alternating sum of the number of vertices, edges and faces.

https://commons.wikimedia.org/wiki/File:Kampsbrezel.jpg

Input

- One line with three space-separated integers $3 \leq V \leq 10^9$, $3 \leq E \leq 10^9$ and $3 \leq F \leq 10^9$.

Output

- A natural number g, the number of holes in the pieces of bread.

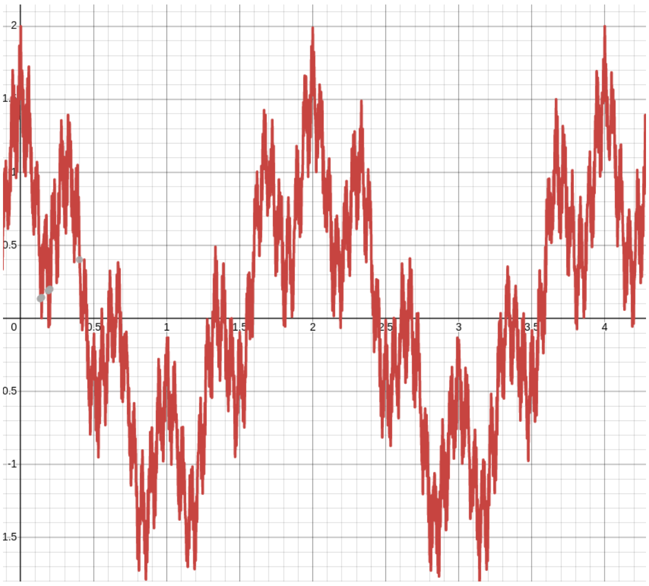
Example

Input	Output
4 6 4	0

Problem 2: Weierstrass Fixed Point

Difficulty: ★ ★ ☆ ☆ ☆
Keywords: Fixed point, Weierstrass function

Let us consider the Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ with $0 < a < 0.9$ and b an odd positive integer such that $ab > 1 + \frac{2}{3}\pi$. In particular, recall this function is continuous, but nowhere differentiable. For a given a, b, we would like to know a fixed point of f with a precision of 10^{-4}.



Graph of a Weierstrass function.

Input

- One line with a real number $0 < a < 0.7$ and an odd positive integer $0 < b < 100$.

Output

- A real number x such that $|f(x) - x| < 10^{-4}$.

Examples

Input	Output
0.5 11	0.4016267206

Problem 3: Dividing the Loot

This problem is based on the ICPC World Finals 2021 Problem C: Fair Division

Difficulty: ★ ★ ★ ☆ ☆
Keywords: Infinite series, Number Theory

There are n pirates that want to divide their loot of m coins in a peculiar way.
The first pirate starts by taking an f fraction of the loot and giving the rest to the next pirate. Then, the second pirate takes of this remaining pile also an f fraction and gives the rest to the next pirate. This process continues. After the nth pirate has taken his fraction, he gives the rest of the pile to the first pirate.

Now as pirates cannot divide one coin, at the end of the infinite process, every pirate wants to end up with an integer number of coins. Given n and m, what is the fraction $f = \frac{p}{q}$ they take out of the pile?
Of course, there are multiple p, q that are possible. Therefore, always return the one with the smallest q value. If there are more values for p that are possible for this certain q, then return the smallest one.

Input

- The integer $6 \leq n \leq 10^4$.
- The integer $0 < m \leq 10^{18}$.

Output

- The integer $p \in \{1, ..., q-1\}$.
- The integer $q > 1$.

Examples

Input	Output
8	1
51000	2

Input	Output
6	2
91000	3

Problem 4: Sturm-Liouville

Difficulty: ★ ★ ★ ☆ ☆
Keywords: Sturm-Liouville problem

Recall a Sturm-Liouville problem is specified by a linear differential operator L. We call a function f a solution of the problem L if $Lf = -\lambda f$, where λ is some real number. Now, let us consider the following Sturm-Liouville problem

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right)P(x) = -\lambda P(x).$$

The solution to this problem on the interval $[-1,1]$ such that $P(1) = 1$, is given by polynomials. The above equation has exactly one polynomial solution of degree d for each non negative integer d. Given a degree d, calculate the coefficients associated to the polynomial with an accuracy of 10^{-6}.

Input

- One line with a non-negative integer $0 \leq d \leq 50$.

Output

- A list of $d+1$ space-separated real numbers that correspond to the coefficients of the desired polynomial of order d in order of increasing degree.

Examples

Input	Output
1	0 1

Input	Output
3	0 -0.6 0 1

Problem 5: Be Squareful

Difficulty: ★ ★ ★ ☆ ☆
Keywords: Number Theory, Squareful numbers

A *squareful number* is a positive integer n such that for any prime p dividing n, p^2 also divides n. For example, 1 and 72 are squareful numbers, but 24 is not. In 1970, Solomon W. Golomb showed that there are infinitely many pairs of consecutive squareful numbers, by noticing that the Pell-equation

$$x^2 - 8y^2 = 1$$

has infinitely many integral solutions. However, this method does not yield all possible pairs of consecutive squareful numbers, such as the pair $(675, 676)$.

We study the differences of squareful numbers in a different manner. Instead of showing there are infinitely many pairs of squareful numbers which differ by 1, we look for the smallest pair of squareful numbers with some given difference k. For instance, the smallest pair of squareful numbers with difference 1 is $(8, 9)$.

Input

- One line with a single integer $1 \leq k \leq 10^5$. It is guaranteed that there is at least one pair of squareful numbers less than $2 \cdot 10^6$ that differ by k.

Output

- The minimal positive integer a such that both a and $a + k$ are squareful numbers.

Example

Input	Output
1	8

Input	Output
6	214369

Input	Output
7	1

Problem 6: Mischievous Moles

Difficulty: ★ ★ ★ ★ ☆
Keywords: Stochastic Process, Graph Theory

Famous Dutch carrot farmer Harold is the reigning champion of the yearly World Carrot Championships. However, in a petty attempt to sabotage Harold in this year's edition, a different carrot farmer has hired a gang of moles under the leadership of the infamous Mr. Mole to mess up Harold's carrot field. This is a big deal, as the layout of a carrot field greatly influences the final score.

As Harold is no stranger to doing business with Mr. Mole, he knows precisely how the gang of moles is structured. Besides Mr. Mole, the gang consists of k other moles. The carrot field consists of n carrot spots, and each mole inhabits a single tunnel between two carrot spots in the field. The task of these moles is to swap the two carrots at both endpoints of their tunnel. In particular, every minute, Mr. Mole selects one of his k moles uniformly at random, and orders it to swap the two carrots at the endpoints of their tunnel.

From prior experience, Harold knows that trying to stop Mr. Mole and his gang is futile. However, Harold does have a significant advantage; being the reigning champion, he has the privilege of being able to call the jury in at any time. Harold therefore decides to call in the jury immediately when the carrots are randomly positioned correctly again by the moles. In particular, Harold positions his n carrots in the optimal arrangement initially, and calls the jury the next time the carrots are in this arrangement again. This can however take some time, so Harold would like to know in advance how long he should expect to wait.

Input

- The first line contains a single integer $2 \leq n \leq 10^6$, the number of carrots Harold has.
- The second line contains a single integer $1 \leq k \leq 10^6$, the number of moles Mr. Mole has.
- Next follow k lines describing the different moles, each containing two space-separated integers $1 \leq a_i, b_i \leq n$, the two endpoints of the tunnel of the i-th mole.

Output

- The expected time T (in minutes) that Harold should wait for the initial arrangement to appear again for the first time. Note that T is always an integer. As T can be quite large, output the number modulo $10^9 + 7$.

Example

Input	Output
2 1 1 2	2

Input	Output
16 15 3 4 1 2 11 7 3 1 2 16 6 4 7 15 8 9 11 12 1 14 16 15 13 8 6 1 5 8 15 10	496038323