Problem : Quantum Quiz - Solution

Recall the definition of an orthoframe. An orthoframe consists of a set of worlds I and a reflexive symmetric accessibility relation $\not\perp$ on I, whose negation we denote by $\perp$. Now, the complement of $X \subseteq I$ is defined as

$$X' = \{i \in I \mid \forall j \in X : i \perp j\}.$$

And $X \subseteq I$ is called a proposition if

$$i \in X \iff \forall j \in I : (i \not\perp j \implies j \not\perp X).$$

Note the right hand side is by contraposition actually the same as $\forall j \in X' : i \perp j$. Then, we see that the forward implication is always true, since if $i \in X$, then by definition of X', we have $i \perp j$ for all $j \in X'$. Thus, to prove something is a proposition it suffices to prove the reverse implication. We have the following lemmas.

Lemma 1. *Let $X \subseteq I$. Then, X' is a proposition.*

Proof. We have to show the reverse implication of the definition of being a proposition. Let $i \in I$ be such that for all $j \in I$, we have $i \not\perp j \implies j \not\perp X'$. That is $j \notin X''$. So, there exists a $k \in X'$ such that $k \not\perp j$. Now, note that as $k \in X'$, $k \perp x$ for all $x \in X$. So, $j \notin X$. So, we have that for all $j \in I$, $i \not\perp j \implies j \notin X$. Taking the contrapositive of this statement, we get $\forall j \in X, i \perp j$. That is $i \in X'$, as desired. ☐

Lemma 2. *Suppose $X \subseteq Y \subseteq I$. Then, $Y' \subseteq X'$.*

Proof. Let $i \in Y'$. Then, for all $y \in Y$, $i \perp y$. In particular, as $X \subseteq Y$, we have for all $x \in X$, $i \perp x$. So, $i \in X'$. ☐

Lemma 3. *For all $X \subseteq I$, we have $X \subseteq X''$. And if X is a proposition, the reverse inclusion also holds. That is by lemma 1, X is a proposition if and only if $X = X''$.*

Proof. Let $i \in X$. Then, note that for all $j \in X'$, we have $i \perp j$. So, in particular, $i \in X''$.
On the other hand, suppose X is a proposition. Then, let $i \in X''$. Then, for all $j \in X'$, we have $i \perp j$. Taking the contraposition of the implication $j \in X' \implies i \perp j$, we get $\forall j \in I : i \not\perp j \implies j \notin X'$. Now, as $j \notin X'$ is the same as $j \not\perp X$, we see that as X is a proposition, $i \in X$. ☐

Proposition 4. *For $X \subseteq I$, the minimal proposition containing X is given by X''.*

Proof. Note that by lemma 1, X'' is a proposition and by lemma 3, X'' contains X. Now, let P be a proposition such that $X \subseteq P$. Then, by lemma 2 and 3, we have $X'' \subseteq P'' = P$. So, X'' is indeed minimal. ☐

Now, computationally we can represent the reflexive symmetric relation $\not\perp$ by a non-directed graph where every vertex has a self edge. Then, for a collection of vertices X, we can compute X' as the set of all vertices not connected to X. This can be done linearly in the number of edges using sets for example by iterating over all $x \in X$ and marking their neighbours as not being in X'. Then, all unmarked nodes are in X'. This runs in $O(n + m)$ time. Applying this operation twice, we obtain X'' the minimal proposition that contains X in $O(n + m)$, which is fast enough.