Problem : Rational Refill - Solution

We look for a non-negative rational solution to $x^3 + y^3 = C := a^3 + b^3$ different from a, b, where a, b are given. First note that this is impossible if $a = 0$ or $b = 0$, due to Fermat's last theorem. Secondly, one can show that it is also impossible if $a = b$. This is difficult to show, but the sample case $1, 1$ is impossible, and thus so is any other case where $a = b$ by scaling. In all other cases, there are other solutions, which can be found with the following method.

We claim that if (z, w) is a solution to $z^3 + w^3 = C$, then so is

$$\left(\frac{z\left(z^3 + 2w^3\right)}{z^3 - w^3}, \frac{-w\left(2z^3 + w^3\right)}{z^3 - w^3} \right).$$

Assuming this holds, it suffices to start with $(z, w) = (a, b)$ and to keep iterating until we reach another non-negative solution. One can show that the denominator keeps increasing so that this process will not yield the original solution. Furthermore, this process will terminate for all possible inputs.

The main question is how one can derive this transformation without pure guessing. The idea comes from the theory of elliptic curves. We consider the curve $x^3 + y^3 = C$. From elliptic curves, if we have two rational solutions on this curve, the line through these two points if it intersects the curve in a third point, this third point is also rational. In the limit of this result, choosing the two initial points as the same point, the same holds for the tangent in a rational point. For (z, w), the second intersection of the tangent at (z, w) with the curve is precisely

$$\left(\frac{z\left(z^3 + 2w^3\right)}{z^3 - w^3}, \frac{-w\left(2z^3 + w^3\right)}{z^3 - w^3} \right).$$

As the gradient at (z, w) is $(3z^2, 3w^2)$, the tangent is given by $3z^2(x - z) + 3w^2(y - w) = 0$. On the other hand we also have $(x^2 + xz + z^2)(x - z) + (y^2 + yw + w^2)(y - w) = x^3 + y^3 - z^3 - w^3 = C - C = 0$. It follows that

$$\frac{z^2}{w^2} = -\frac{y - w}{x - z} = \frac{x^2 + xz + z^2}{y^2 + yw + w^2}.$$

Hence

$$0 = z^2(y^2 + yw + w^2) - w^2(x^2 + xz + z^2) = y^2z^2 + yz^2w - x^2w^2 - xzw^2 = (yz - xw)(yz + xw + zw).$$

Now, if $yz - xw = 0$ then (x, y) would be a rational multiple of (z, w), but as $x^3 + y^3 = C = z^3 + w^3$ this would imply $(x, y) = (z, w)$. We however choose the other intersection point, so we have $yz + xw + zw = 0$, that is $(y - w)z + (x - z)w + 3zw = 0$. Combining with the tangent formula $z^2(x - z) + w^2(y - w) = 0$, which are both linear formulas, we find unique solutions for x and y. These correspond to precisely the point described above.